19

ELECTROMAGNETIC WAVES COLLIDING WITH GRAVITATIONAL WAVES

In the previous chapters we have considered the collision between two electromagnetic waves, permitting them also to be coupled to gravitational waves. In this chapter a situation is considered in which a plane gravitational wave approaches and collides with a plane electromagnetic wave. In this situation, the field equations are identical to those of previous chapters but the initial conditions are different.

19.1 A simple example

A particularly simple example is the case when an impulsive gravitational wave described by the component $\Psi_4 = a\delta(u)$ collides with a step electromagnetic wave described by $\Phi_0 = b\Theta(v)$. An exact solution describing this case has been given by the author (Griffiths 1975*b*). In this case, the line elements describing the initial regions are given by

$$\begin{aligned}
&\text{Region I}: && \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 \\
&\text{Region II}: && \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1-au)^2\mathrm{d}x^2 - (1+au)^2\mathrm{d}y^2 \\
&\text{Region III}: && \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 bv(\mathrm{d}x^2 + \mathrm{d}y^2).
\end{aligned} \tag{19.1}$$

In terms of the metric functions of the Szekeres line element (6.20), the solution of the field equations (6.21-22) for the interaction region IV, satisfying the required boundary conditions is given by

$$\begin{aligned}
e^{-U} &= \cos^2 bv - a^2u^2, & e^V &= \frac{(1-au)}{(1+au)}, \\
e^{-M} &= \frac{\cos bv\sqrt{1-a^2u^2}}{\sqrt{\cos^2 bv - a^2u^2}}, & W &= 0, \\
\Phi_0^\circ &= \frac{b\cos bv}{\sqrt{\cos^2 bv - a^2u^2}}, & \Phi_2^\circ &= -\frac{a\sin bv}{(1-a^2u^2)\sqrt{\cos^2 bv - a^2u^2}}.
\end{aligned} \tag{19.2}$$

This solution can be seen to have a number of interesting features. The incoming gravitational and electromagnetic waves are plane and parallel propagating. That is, they follow non-expanding twist-free and

shear-free null geodesic congruences. When the gravitational wave meets the electromagnetic wave it starts to contract. By contrast, the electromagnetic wave both contracts and shears. The contraction of both waves becomes unbounded on the space-like hypersurface $\cos^2 bv = a^2u^2$ which can be seen to correspond to a scalar polynomial curvature singularity. These properties are as expected according to the discussion in Chapter 5.

It can be seen from (19.2) that the electromagnetic wave is partially reflected on collision with the gravitational wave. This occurs because the congruence along which the electromagnetic wave propagates is focused astigmatically by the gravitational wave, and is no longer shear-free. According to the Mariot–Robinson theorem 5.2, null electromagnetic waves necessarily propagate along shear-free null geodesics. It follows that the electromagnetic field in the interaction region cannot remain null and therefore some back scattering must occur.

This particular feature has also been confirmed by Sbytov (1973), who considered the propagation of a test electromagnetic wave through a plane gravitational wave.

Other interesting features of this solution can be seen after first evaluating the gravitational wave components. Using (6.23), the only non-zero components in the interaction region are given by

$$\begin{aligned} \Psi_4^\circ &= a\delta(u) - \frac{3a^3u\sin^2 bv}{(1-a^2u^2)^2(\cos^2 bv - a^2u^2)} \\ \Psi_2^\circ &= -\frac{a^2b\,u\sin bv\cos bv}{(\cos^2 bv - a^2u^2)^2}. \end{aligned} \tag{19.3}$$

From the absence of a Ψ_0 component, it can be seen that, unlike the electromagnetic wave, the gravitational wave is not partially reflected. However, the usual coulomb component Ψ_2 is still generated by the collision, and the component Ψ_4 develops a tail.

It follows from the absence of the Ψ_0 and Ψ_1 terms that the gravitational field in the interaction region, as represented by the Weyl tensor, is of algebraic type II. However, in this situation, the repeated principal null congruence of the gravitational field is not aligned with a principal null congruence of the electromagnetic field.

Another simple solution has been given elsewhere (Griffiths 1976*b*). This has the same general properties as the above solution, but the initial impulsive gravitational wave has been replaced by a step wave with the line element (4.21).

19.2 General initial data

Consider now the situation of a totally general collision between an arbitrary gravitational wave and an arbitrary electromagnetic wave. It is

convenient to assume that the gravitational wave approaches in region II, and the electromagnetic wave in region III. The background in region I is assumed to be flat with metric (3.6).

Region II is now considered to contain a general gravitational wave. The appropriate line element is (4.13). This may be compared with the Szekeres line element (6.20) with the condition that $V = V(u)$ and $W = W(u)$. Also $e^{-U} = (\frac{1}{2} - f)$ where $f = f(u)$, and it is possible to scale the coordinate u such that $e^{-M} = 1$ giving

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (\tfrac{1}{2}+f)\big(e^V \cosh W \mathrm{d}x^2 - 2\sinh W \mathrm{d}x\mathrm{d}y + e^{-V}\cosh W \mathrm{d}y^2\big). \tag{19.4}$$

This gravitational wave is completely arbitrary and is described in terms of three functions $f(u)$, $V(u)$ and $W(u)$ that are constrained only by the single equation

$$\frac{2f''}{(\frac{1}{2}+f)} - \frac{f'^2}{(\frac{1}{2}+f)^2} + W'^2 + V'^2\cosh^2 W = 0 \tag{19.5}$$

which is equivalent to (6.22c). From (6.23) it can be seen that the approaching gravitational wave is given by the single component

$$\begin{aligned}\Psi_4(\textsc{ii}) = &-\tfrac{1}{2}(V''\cosh W + iW'') - V'W'\sinh W + \tfrac{1}{2}iV'^2\sinh W\cosh W\\ &- \frac{f'}{2(\frac{1}{2}+f)}(V'\cosh W + iW').\end{aligned} \tag{19.6}$$

In order to satisfy the appropriate boundary conditions across the wave front, it is also necessary to assume that

$$f = \tfrac{1}{2}, \qquad V = W = f' = V' = W' = 0, \qquad \text{when} \qquad u = 0. \tag{19.7}$$

Region III here may be considered to contain a general plane electromagnetic wave. It is assumed that there is no associated free gravitational wave, so that the metric is asymptotically flat. It turns out to be convenient to take the line element in this region in the form

$$\mathrm{d}s^2 = 2e^{-M}\mathrm{d}u\mathrm{d}v - (\tfrac{1}{2}+g)(\mathrm{d}x^2 + \mathrm{d}y^2) \tag{19.8}$$

where $g = g(v)$ and $M = M(v)$. The electromagnetic wave is described by the component Φ_0. The amplitude of this wave is determined by the equation (6.22b) which becomes

$$4\Phi_0^\circ\bar{\Phi}_0^\circ = -\frac{2g''}{(\frac{1}{2}+g)} + \frac{g'^2}{(\frac{1}{2}+g)^2} - \frac{2g'M'}{(\frac{1}{2}+g)}. \tag{19.9}$$

The phase of the wave, however, is completely arbitrary, as is required by the fact that a plane electromagnetic wave is only determined by the metric up to an arbitrary duality rotation.

It is, of course, possible to rescale the null coordinate v in order to put $M=0$. This would leave the electromagnetic wave being described by the single function $g(u)$ and an arbitrary phase. In order to ease the integration of the field equations in region IV, however, it is preferable to retain the additional function, and to regard $g(v)$, $\Phi_0(v)$ and $M(v)$ as arbitrary functions that, in region III, are required to satisfy (19.9).

In order to satisfy the junction conditions across the boundary between regions I and III, it is assumed that

$$g=\tfrac{1}{2}, \qquad M=g'=M'=0, \qquad \text{when} \qquad v=0. \tag{19.10}$$

The initial conditions that have now been set describe the collision between a completely general gravitational wave and a completely general pure electromagnetic wave. The corresponding general solution describing the subsequent interaction has not yet been obtained. However, one particular class can easily be obtained and will be described in the next section.

19.3 A general class of solutions

As frequently described in previous chapters, the line element in the interaction region may be considered in the Szekeres form (6.20), and the field equations are (6.21) and (6.22).

As in (6.24), it is always possible to integrate (6.22a) to obtain

$$e^{-U}=f(u)+g(v) \tag{19.11}$$

where, to satisfy the boundary conditions, $f(u)$ and $g(v)$ necessarily take the same form as they have in regions II and III respectively. These functions are thus exactly of the form that is specified by the approaching waves.

A general class of solutions (Griffiths 1983) has previously been obtained in which the metric functions V and W are assumed to be independent of v. In this case

$$V=V(u) \qquad \text{and} \qquad W=W(u) \tag{19.12}$$

take exactly the same form in the interaction region as they do in region II. This class of solutions is thus almost completely determined by the functions $f(u)$, $V(u)$, $W(u)$ and $g(v)$ that are specified by the approaching waves.

With the condition (19.12), the field equations (6.22d) and (6.22e) imply that

$$\Phi_0^\circ \bar{\Phi}_2^\circ = \frac{g'(V'\cosh W - iW')}{4(f+g)}. \tag{19.13}$$

With this condition, Maxwell's equations (6.21) then imply that

$$\begin{aligned} \Phi_0^\circ &= \frac{-g'}{2\sqrt{f+g}\sqrt{\frac{1}{2}-g}}\, e^{i\alpha(u)} \\ \Phi_2^\circ &= -\frac{\sqrt{\frac{1}{2}-g}}{2\sqrt{f+g}}(V'\cosh W + iW')\, e^{i\alpha(u)} \end{aligned} \tag{19.14}$$

where α is now a function of u only, and is required to satisfy the equation

$$\alpha' = -\tfrac{1}{2}V'\sinh W. \tag{19.15}$$

The electromagnetic field is now determined completely by the given functions up to an arbitrary constant phase. Finally, the remaining equations in (6.22) can be integrated to give

$$e^{-M} = \frac{-g'\sqrt{\frac{1}{2}+f}}{2a\sqrt{f+g}\sqrt{\frac{1}{2}-g}} \tag{19.16}$$

where a is an arbitrary constant.

The metric functions and the electromagnetic field components in the interaction region are now all determined. The components of the Weyl tensor can be obtained from (6.23) and, using (19.6), the non-zero components can conveniently be written in the form

$$\begin{aligned} \Psi_4^\circ(\textsc{iv}) &= \Psi_4(\textsc{ii}) - \frac{3f'(\frac{1}{2}-g)}{4(f+g)(\frac{1}{2}+f)}(V'\cosh W + iW') \\ \Psi_2^\circ(\textsc{iv}) &= -\frac{f'g'}{4(f+g)^2}. \end{aligned} \tag{19.17}$$

Two fundamental properties of this solution can immediately be deduced from these expressions.

Firstly, it may be observed that these solutions necessarily contain a scalar polynomial curvature singularity in region IV on the space-like surface given by $f+g=0$. This is common to most colliding plane wave solutions.

Secondly, it may be observed that the gravitational wave is algebraically special and is of algebraic type II. The congruence on which $v =$ const is a repeated principal null congruence of the gravitational field, but is not a principal null congruence of the electromagnetic field.

This class of algebraically special solutions thus contains a non-aligned non-null electromagnetic field. Its relation with other non-aligned non-null Einstein–Maxwell fields has been investigated elsewhere (Griffiths 1986), where a more general class of such fields has been obtained. This more general class contains two distinct sub-classes, one of which is the class obtained here, while the other contains all other known solutions of this type.

Another important property of this class of solutions can be deduced from the electromagnetic field components given by (19.14). It can immediately be seen that the electromagnetic wave has been partially reflected by the gravitational wave. This property has already been noted in the special case described in Section 19.1.

It may also be noticed that the expression for M given by (19.16) is only continuous across the boundary between regions III and IV if, in region III, M is given by

$$e^{-M} = \frac{-g'}{2a\sqrt{\frac{1}{2}-g}\sqrt{\frac{1}{2}+g}}. \tag{19.18}$$

where a is a constant. The other boundary conditions (19.10) are then only satisfied if g has the form

$$g = \tfrac{1}{2} - a^2v^2 + \ldots \tag{19.19}$$

The approaching electromagnetic wave is then given by

$$\Phi_0^\circ(\text{III}) = \frac{-g'}{2\sqrt{\frac{1}{2}-g}\sqrt{\frac{1}{2}+g}}e^{i\alpha} \tag{19.20}$$

where α is a constant. In this case, a coordinate transformation

$$v \to \tilde{v}, \qquad \text{where} \qquad g = \tfrac{1}{2} - \sin^2 a\tilde{v} \tag{19.21}$$

can be used to put $M = 0$, and it can then be seen that the approaching wave is necessarily the step wave given by

$$\Phi_0(\text{III}) = ae^{i\alpha}\Theta(\tilde{v}) \tag{19.22}$$

where the arbitrary constant phase is of no significance.

Since M and g are now completely determined with $M = 1$ in region III, it follows that the electromagnetic field component is also completely determined. Thus, although this solution includes a completely general gravitational wave in region II, the approaching electromagnetic wave can only be the step wave given by (19.22).

Finally, it is of interest to re-express this class of solution in terms of the notation of the previous chapters. With the components (19.14), equations (16.4) can be integrated to give the potential function H in the form

$$H = \sqrt{\tfrac{1}{2} - g}\, e^{V/2} \sqrt{1 + i \sinh W}\, e^{i\alpha}. \qquad (19.23)$$

In view of (19.12) and the definitions (11.2) and (11.3), it is clear that χ and ω are here functions of u only. However, they may subsequently be considered as functions of f only or, using (10.9) (see also the appendix), as functions of $(\psi + \lambda)$ only:

$$\chi = \chi(\psi + \lambda), \qquad \omega = \omega(\psi + \lambda). \qquad (19.24)$$

This approach, however, does not lead to simple expressions for Ψ, Φ and Z defined respectively by (16.14), (16.13) and (16.19).